Schemes 2018 Exercise 3

Question 1. Let $A \subseteq B$ be rings such that B is integral over A.

- Let $\mathfrak{b}$ be an ideal in B. Prove that $B/\mathfrak{b}$ is integral over $A/(A \cap \mathfrak{b})$
- For every multiplicative system $S \subseteq A$, $S^{-1}B$ is integral over $S^{-1}A$.

Question 2. Let $A \subseteq B$ and let C be the integral closure of A in B. Let S be a mult. system in A. Prove that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

Question 3. Prove that a UFD (unique factorization domain) is integrally closed. i.e. it is integrally closed in its field of fractions.

A property P of rings is called **local** if TFAE:

- A has P.
- $A_{\mathfrak{p}}$ has P for every prime ideal $\mathfrak{p}$.
- $A_{\mathfrak{m}}$ has P for every maximal ideal $\mathfrak{m}$.

Question 4. Let A be an integral domain. Prove that being integrally closed is a local property for A.

Question 5.

- Let C be the integral closure of A in B. Then the integral closure of an ideal $\mathfrak{a} \subseteq A$ in B is $\sqrt{\mathfrak{a}C}$.
- Let $A \subseteq B$ be integral domains, with A integrally closed. Let $\mathfrak{a} \subseteq A$ be an ideal and $x \in B$ integral over $\mathfrak{a}$. Then if $t^n + ... + a_0$ is the minimal monic poly. for x over $A_{\{0\}}$ then $a_i \in \sqrt{a}$.
- Prove the "going down theorem": Let A be integrally closed and $A \subseteq B$ integral over A. Let $\mathfrak{p}_1 \supset ... \supset \mathfrak{p}_\mathfrak{n}$ be prime ideals in A. Let $\mathfrak{q}_1 \supset ... \supset \mathfrak{q}_\mathfrak{m}$ be a sequence lifting $\mathfrak{p}_i$ to B, $i = 1, ..., m$. Then it can be completed to a lifting of all the $\mathfrak{p}_i$.

Question 6. * Let $R = \mathbb{C}[x, y]/(x^2(x + 1) - y^2)$.

- Find the integral closure of this ring in its field of fractions.
- Denote the integral closure of R by $\tilde{R}$. Let $i : R \to \tilde{R}$ be the inclusion. Find all the closed points for which $(i^*)^{-1}(x)$ contains more then one point.
- *Draw the set of solutions to the equation $x^2(x + 1) = y^2$ (i.e. the real points). What is special about the points you found in the previous bullet? (this is not a precise mathematical question, so I don't expect precise answer!)